THE PARADOXES OF MR. RUSSELL

WITH A BRIEF ACCOUNT OF THEIR HISTORY

BY

EDWIN R. GUTHRIE, JR.

A THESIS

PRESENTED TO THE FACULTY AND TRUSTEES OF THE UNIVERSITY OF PENNSYLVANIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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In his *Principles of Mathematics* published in 1903 Mr. Bertrand Russell found himself involved in a difficulty in his theory of classes for which he could at that time see no solution. This difficulty lay in a series of paradoxes which were in one form or another much discussed in Scholastic logic but which had fallen into disrepute and had been little studied until the recent development of symbolic logic brought them again to notice. In the recent discussion the paradoxes have presented themselves in a form somewhat different from the *insolubilia* which occupied so much space in Scholastic text-books, but they are essentially the same and the Scholastic attempts at solution bear many points of resemblance to the principal solutions offered by the writers who have worked over the difficulty since Russell's rediscovery of it.

The paradoxes mentioned by Russell are the first seven of the following list, to which have been added a number of others of the same general character.

1. The first is the classic example of the Cretan Epimenides who says, according to St. Paul, that all Cretans are liars, as well as other things that have nothing to do with the case,—and by this statement makes himself a liar.

2. Represent the class of all classes that are not members of themselves by w. Then $(x \in w) < (x \in x)'$. Now let x have the value w; we get: $(w \in w) < (w \in w)'$.[1]

3. Let T be the relation that exists between R and S whenever R does not have the relation R to S. Then $RTS = (RRS)'$. Making R and S both T, we have $TTT = (TTT)'$.

[1] (ϵ) in Russell's notation indicates the membership of an individual in a class.

4. The next contradiction is one given by Burali-Forti. Any series of ordinals beginning with 0 has an ordinal number one greater than the highest term. The series of all ordinals would then have an ordinal number greater than the highest ordinal.

5. "The least integer not namable in fewer than nineteen syllables" (which Russell states to be 111,777) has been named in eighteen.

6. All finite combinations of the letters of the alphabet can be ranged[1] so as to be in one to one correspondence with the series of ordinals, so that the total number of definitions is *aleph,* the first transfinite number. The number of transfinite ordinals exceeds this so that there are indefinable ordinals, and among these there is a least. This has, however, been defined.

7. The paradox of Richard is: Let E be the class of all decimals that can be defined in a finite number of words. Its terms can be ordered as were the definitions in (6). Then define N so that the nth figure of the nth decimal of N is one greater than the nth figure of the nth decimal of E. N is then defined in a finite number of words, but still is not a member of E.

For convenience of reference we may state here several other paradoxes which have been mentioned in the discussion. The next is due to Russell:

8. Every concept can be predicated of itself or it cannot. If it can, let it be called *predicable,* if not, *impredicable.* Then the concept *impredicable,* if predicable is impredicable, and if impredicable is predicable.

9. This proposition is false.

10. All propositions are false.

11. I lie.

[1] This arrangement Gorsch has shown in his *Mengenlehre,* Abhandlungen der Fries'chen Schule, I: 508. The combinations one at a time, then two at a time, then three, and so on, can be given in a definite order.

Mr. Alexander Rüstow has collected a large number of references to similar paradoxes in ancient writings, notably in the works of Aristotle, Plato, and Diogenes Laertius.[1]

Among these ancient writers the paradox of the Liar was the one to attract the most attention. As a rule the paradox was accepted as final, and the fact that it existed was used in support of an attack on the validity of human knowledge, as Montaigne used it later on. What solutions were proposed were crude attempts to place it under one of the Aristotelian forms of fallacy. No analysis of the paradoxes was made, nor were they recognized as a class. It was not until the time of the Scholastic logicians that they were presented in a form which offers interesting parallels to our present statement and analysis.

During the Scholastic period the interest which the paradoxes or Insolubilia aroused was so great that many of the text-books of logic written from the fourteenth to the sixteenth centuries devoted lengthy chapters to them and there were a number of separate treatises in which their solution was attempted.

Anything like a careful analysis of the Scholastic doctrines concerning the Insolubilia would require an extensive knowledge of medieval logic and might not be justified in a study of the logical side of the problem. But it is of some interest to see the form in which the difficulties arose and the types of solution offered, and in particular the close analogies which these bear to the solutions of more modern writers.

The first collections of Insolubilia contained a number of

[1] Mr. Rüstow's thesis consists of an extended examination of ancient literature for references to the paradox of the "Liar" and a discussion of the uses to which the fact of paradox was put. There is also a short list of sentences from Scholastic writers giving the substance of their solutions. He has stated the paradoxes in two type forms that are very enlightening, and with these statements as a basis he offers a solution of his own which will come in for a share of our consideration. *Der Lügner*, Leipsig, 1910.

paradoxes which had in common only the characteristic that they were difficult of solution. It came gradually to be recognized that there was a class of these that contained a peculiar difficulty and were not to be attacked by ordinary methods. These were the propositions which purported to apply to themselves and to state directly or indirectly their own falsity.

One of the longest of these lists, one compiled after this distinction had been recognized, was that of Albertus de Saxonia who wrote in Vienna before 1390. Four of his examples are typical of the rest. These are:[1]

1. *Ego dico falsum.*

2. *Haec propositio est falsa.*

3. *Ponatur, quod Socrates dicat illam, "Plato dicit falsum," et Plato dicat illam, "Socrates dicit verum."*

4. *Si deus est, aliqua conditionalis est falsa, et sit nulla alia conditionalis.*

The first of these reduce to the type of "This proposition is false," and the last makes its own truth imply the falsity of a class of propositions to which it itself belongs, as does "All propositions are false." The rest of his long list for the most part reduce to these two types, intrigued by the use of terms whose meaning is not always definite; for example, "*posito quod in mente Socratis sit ista, 'Socrates decipitur,' et nulla alia, et Socrates credat illam esse veram, quaeritur, an Socrates credendo eam esse veram, decipiatur.*"

Of the solutions proposed for these two forms of the paradoxes Prantl has mentioned the most important and these have been compiled by Rüstow. Among the whole number some are obscure and are dependent on distinctions and rules which belong to the complicated apparatus of medieval logic

[1] For this period Prantl has collected a great number of passages dealing with the Insolubilia. The only important omissions that I have been able to find are Wycliffe's Solution in his *Tractatus de Logica* of which Prantl makes no mention, and an incomplete representation of the solution of Paulus Venetus.

which modern logic has discarded. But the majority make use of comparatively simple devices.

The first solution seems to have been that of Buridan, written in the fourteenth century. One of the premises which leads to the paradox is the law of excluded middle. In the Insolubilia we have examples of propositions which are both true and false. Therefore the law must be rejected.[1]

Another attempt to deal with the situation is made by Hentisberus and mentioned by Buridan.[2] This is based on the doctrine of *restrictio* which was worked out in great detail by the Scholastics. This doctrine states that the denotation of a term, which as far as the explicit statement goes might be taken to be general, is often limited by the context,—an approximation to the more modern universe of discourse. In the case of the Insolubilia this unexpressed restriction limits the denotation of a term in a proposition whose verb is in the present tense to time immediately preceding the present instant, with the idea that the time indicated is that at which the proposition is begun, not that in which it is expressed.

This solution has, like the next, a basis in a psychological analysis. We feel that no thought can really be "of itself." The restriction which prevents this can be made in another way. Peter von Ailly in his tractate on the Insolubilia[3] argues that these Insolubilia all lie in the field of verbal or written propositions as distinguished from mental propositions. These written and spoken propositions are not really propositions at all, and are merely the expression of mental propo-

[1] Grelling-Nelson note this possibility, and mention that it is this axiom that is used to define paradox. Abhandlungen der Fries'chen Schule: II.

[2] Prantl, IV: 89.

[3] Prantl, IV: 104. "*Nulla propositio mentalis proprie dicta potest significare se ipsam esse falsam. . . . Impossibile est intellectui primo formare propositionem universalem mentalem proprie dictam significantem omnem propositionem mentalem esse falsam. . . . Nulla propositio mentalis proprie dicta potest significare, se ipsam esse veram, . . . nec potest habere reflexionem supra se ipsam.*"

Cf. also Occam, quoted by Johannis Majoris (Pr. IV: 250). A *simpliciter insolubile* is not possible.

sitions. It is to the last only that the predicates true and false apply. Furthermore, no mental proposition can really denote itself or have itself as one of its terms, nor can any term represent itself "formally." This would represent an impossible state of mind.

The rule for avoiding the paradoxes becomes, "*Pars propositionis non potest supponere pro toto.*"

This method of dealing with the problem appealed to a number of later writers, Johannis Majoris Scotus, Olkot, and Rosetus among them. It is much the same as Russell's device of the theory of types which depends on the principle of the vicious circle, namely that no term in a proposition can presuppose the proposition or have it as one of its possible values.

The only Scholastic to make a serious criticism of this view was Wycliffe[1] and the difficulty which he points out is much the same as that contained in the theory of types.[2] If we are not to allow a proposition to refer to itself we make a general proposition like "All propositions are true or false" exceptive. It becomes, "All propositions are true or false except this proposition." We would seem to do away with all general propositions about propositions and there are some of these which we do not wish to reject. Wycliffe's criticism takes for consideration the proposition, "A is A." This would not be true universally, for there is one value of A, namely the proposition itself, to which it could not apply.

Wycliffe's own solution lies in denying that the paradoxical statement, "This proposition is false," is true or false in one sense of truth and falsehood, and so denying that it is a proposition at all. The criterion of truth is correct representation of a situation independent of the proposition itself, and in the given instance there is no such independent situation represented. He does not go on to consider what the effect of

[1] Tractatus de logica, p. 197.

[2] This criticism parallels that of Russell by Grelling-Nelson.

this theory would be on "All propositions are true or false" where there is both independent reference and self reference.[1]

Wycliffe's theory is really of the type of another theory termed by an unknown author of a Paris manuscript[2] *cassatio* which involved denying that the propositions in question were propositions at all. Since this theory was not accompanied by a definition of proposition which would exclude these propositions only it was only a beginning of a solution.

Besides the theory of restriction the theory which found the most support was that of Paulus Venetus, of a type which we shall have occasion to refer to later as null-class solution. This pointed out that every proposition implies its own truth, and that the propositions in question assert also their own falsity. Any proposition thus implying contradictories is false, and these propositions are then false. Paulus states this concisely.[3] "*Hoc est falsum significat quod hoc est falsum et quod hoc est verum, sed quod hoc sit falsum et hoc sit verum est impossibile simpliciter.*" This is like the comment of Mansell's Aldrich, that these sentences are merely contradictory and so say nothing.[4]

I have chosen only a few of the Scholastic doctrines for mention for there were many which did not represent anything more than a rhetorical treatment like that of the Scot David

[1] This is like the solution proposed in Mansell's Aldrich. "*Incipiat Socrates sic loqui, 'Socrates dicit falsum' et nihil amplius loquatur: tum interrogat aliquis utrum vera an falsa sit haec propositio. Respondeo, nec veram nec falsam esse, sed nihil significare, nisi aliquid aliud respiciat, quod a Socrate ante dictum supponitur. Qui enim profert haec verba, 'Socrates dicit falsum,' fert judicium de dicto Socratis: quique fert judicium, necessario praesupponit aliquid de quo judicet: unde cum sententia praesupponat objectum suum, clarum est eandem numero propositionem, et sententiam et eius objectum esse non posse. Quare et Scholarum subtilitas hic nihil profecit; nihilque opus est plura dicere de Insolubilibus.*

[2] Prantl, IV: 41.

[3] Logica Pauli Veneti, Venetiis, 1654, p. 84.

[4] "*Sed qui ut verum simul dicat et mentiatur dicit unum aliquid, cuius partes sibi invicem contradicunt, is nec verum, nec falsum, sed omnino nihil dicat; quando enim sententiae pars una evertit alteram, tota nihil prorsus significat sed inaniter strepit.*"

Cranston who dismissed them with the rule that no proposition really falsified itself when the contrary could be rationally maintained.

After the decline of the Scholastic logic the problem of the Insolubilia fell into disrepute. The only mention of it is the expression of a wonder that men could have been inclined to discuss so subtle and vain a question. This attitude found occasional expression until modern times along with a contempt for all purely formal difficulties. Even in the last century Lotze dismisses the problem with the remark that when an assertion involves something in regard to the fact of its assertion which makes the assertion impossible or untrue, it is formally insoluble, and he does not seem to feel any desire to change this state of affairs. He seems to think that the essential difficulty is overcome in pointing out that we can say "I lied" instead of "I lie" and there will be no contradiction.

In 1869 Mr. C. S. Peirce published an article on "The Validity of the Laws of Logic" in which he considered the paradox of the proposition, "This proposition is false."[1] Although Mr. Peirce was one of the leaders in opening the field of symbolic logic his discussion of the Insolubilia has more in common with the Scholastic logicians than with the modern. He himself calls attention to the fact that his solution is identical with that of Paulus Venetus except that he offers a proof for a principle which Paulus assumes to be true.

Mr. Peirce shows that the proposition, "This proposition is false" whether it be assumed false or true can be proved both false and true by regular procedure. But an examination of the reasoning which leads from the assumption that the proposition is false to the conclusion that it is true shows that it is based on a false premise, namely that all that the proposition says is that it is false, whereas every proposition

[1] *Journal of Speculative Philosophy*, II: no. 4.

implies in addition to this that it is true. Hence, though the proposition is false, that does not make it true or make it assert the truth, for it asserts that it is both false and true. This solution we may call a nul-class solution since it lies in showing that the proposition implies contradictories and so is false.

Besides these notices and a comment by the editor of Wycliffe's *Latin Works* in which he suggests that the paradoxical propositions are not really propositions at all, I have been able to find no serious consideration of the paradoxes until the development of symbolic logic brought them to the light in a form which could not be ignored. It was Mr. Russell who first noticed these difficulties in connection with the theory of classes and considered several devices for overcoming them. The one to which he gave preference was the no-class theory and it was from this that his perfected theory of types grew. This theory of types will require an extended statement.

To deal with these paradoxes Russell has invented a theory of types that Mr. Harold Chapman Brown has very sagely characterized as a disguised universe of discourse, but it is rather more than that,—it is a theory of the universe of discourse.

Russell's logic differs from the traditional logic in admitting expressions that he calls propositional functions, or simply functions. These are not propositions but are derived from propositions by making one or more of the terms indefinite. The proposition "Socrates is a man" of the text books can be expressed "x is a man" in which x means that a term is to be supplied. These propositional functions are of two kinds, those in which the truth value of the function depends on the value given the variable and those in which the truth value is the same for all values of the variable. The function "x is a man" is of the first type; "'x is a man' implies 'x is mortal'"

is of the second. Only functions of the second type can be called true or false and treated as propositions in logic. The variable that occurs in functions of this type Russell calls an apparent variable. Examination of the paradoxes has led him to conclude that all of them contain an apparent variable that makes reference to "all" of a totality that is really illegitimate. The apparency of the variable is deceitful. The understanding "for all values" is never quite justified, for there is sometimes possible one value that will result in paradox. This troublesome value is that of the function itself. In the case of "All propositions are false" we may make the statement, "For all values of x, x is a false proposition," and then insert as a value of x this statement itself and the result is paradox. This leads to the conclusion that such variables must have ranges of values that result in significant propositions and that the function itself must be excluded from this range in each case. "Whatever contains an apparent variable must not be one of the possible values of that variable."

This exclusion of a function from the range of significance of a variable occurring in it is accomplished by the theory of types which is really a theory of ranges of significance, or, to use the more familiar term, of universes of discourse. A logical type is defined as the range of significance of a propositional function. The purpose of the theory of types is to rule the function itself out of the range of significance of its variables. A proposition is not one of the things to which it can refer. This exclusion is accomplished by establishing a series of ranges of significance. The first range is defined as including all "individuals" and these are known by the character of lack of complexity, which only means that they are not analyzable into term and predicate as are propositions. These are the first logical type, and the functions in which they occur are first order functions. These first order functions constitute the second logical type, and the functions into which

they enter, the second order of functions, and the process may be extended indefinitely. The purpose of the theory is accomplished by defining functions whose terms are of a type below their own as "predicative functions" and ruling that only predicative functions are to be admitted.

The effect of this on the paradoxes is evident. Where "all propositions" occurs in a proposition of order n the proposition, to be legitimate, must have, understood or expressed, the addition "of order less than n." "I lie" becomes "There is a proposition of order n which I affirm and which is false," and this is of order one greater than n; hence no proposition of order n has been uttered and the statement is simply false.

This is the theory in its essentials and as far as the paradoxes are concerned is enough to meet the situation. When the theory is incorporated into the logic Russell finds an addition necessary, and with this addition we are led into a number of difficulties of serious nature. Incidentally we find ourselves unable to make any statement concerning truth or meaning, or logic and have the statement itself true or meaningful or logical within the meaning of its own terms. But what is of more concern to the mathematical logician is that at this point we find ourselves unable to speak of all the properties of a term, and yet such reference is very necessary at times. It is through such reference that mathematical logic has been in the habit of defining identity. The usual definition is: For all values of φ, $\varphi(x)$ implies $\varphi(y)$, or y has all the properties of x. But this statement itself is a function of x or a property of x, and so is a possible value of x and the definition is not predicative. To make it possible to refer to all the properties of a term the axiom of reducibility is invented, the assumption that there exists for any given function a predicative function that is true when it is true and false when it is false. Applied to the non-predicative function "All propositions are true or false" this would assert that for every set of propositions there will be found a proposition of higher order which will assert

of these that they are true or false, though this proposition will not apply to itself and if it is to be included must be taken to be a part of a new set consisting of the propositions to which it referred, and itself.

The intended effect of the axiom of reducibility is to make the theory of types less "drastic," and drastic it surely is, as its elaborate introduction into Russell's logic shows, but whether it accomplishes this result is not so clear. Through the theory of types we have declared that non-predicative functions are not to be allowed and that they cannot be called true or false. If the predicative function which the axiom assures us accompanies each non-predicative function is to be true only when the non-predicative function is true, then we must have had a true non-predicative function and there could have been no need for the other. All that the axiom can mean is that for every function there exists a predicative function that is true when the given function is meant to be true, and false when it is meant to be false, and this stands in great need of explanation that would have to include among other things a definition of "meant to be true."

The theory of types avoids the paradoxes by declaring them illegitimate propositions and we might accept this outcome as satisfactory if it were not that the theory does not confine its effect to the propositions that give trouble but extends to others in which no contradiction appears, and in some of these the doubtful recourse of the axiom of reducibility is not available as it is for "All propositions are true or false." In "This proposition contains five words" and "This proposition is true" we have sentences that seem to have meaning and yet are declared to be non-predicative. We are forbidden to use them along with the propositions for which the theory was invented.

[1] That the axiom must avail itself of its own protection in order not to be rejected by the principle of the vicious circle presents another curious paradox like the one pointed out in the latter principle by Grelling-Nelson quoted below.

One of the criticisms of Russell's theory brings up an interesting question. K. Grelling and L. Nelson[1] call attention to the fact that the principle of the vicious circle, that no function may make reference to itself in one of its terms, makes a statement concerning all functions and hence concerning itself. They might have gone farther and pointed out that since the antecedent of every implication contains understood all the axioms and principles of our logic, on Russell's view every proposition asserts "I make no reference to myself." When we ask whether our principle that "only predicative functions are to be allowed" is itself predicative, we find ourselves in as serious a difficulty as any of the paradoxes could offer, but it is doubtful whether we can ever require a principle to stand its own test. Many principles of logistic cannot be expressed in the logistic for which they legislate. We cannot express in symbolic language the convention to use Roman type to represent propositions; nor can the right to substitute an expression for its equivalent be symbolically expressed without assuming the right in the expression. With every logic there must be a body of extra-symbolic principles that govern the use of the symbolism.

There has been one other solution for the paradoxes worked out with the care of Mr. Russell's. In 1910 Mr. Alexander Rüstow published a monograph on "The Liar"[2] in which he has collected a number of references to the paradox in ancient writings and made a brief summary of the scholastic solutions given in Prantl's history of logic. His own theory is a development from the analysis of Grelling-Nelson who had stated the paradoxes in a type form in the language of theory of classes.[3]

Rüstow finds that all the paradoxes can be stated in one of two forms, one of them for paradoxes of the type of "This proposition is false" and the other for those akin to "All propositions are false." The first of these is:

[1] Abhandlungen der Fries'chen Schule, II.

[2] *Der Lügner, Theorie, Geschichte und Auflösung*, Leipzig, 1910.

[3] Abhandlungen der Fries'chen Schule, vol. II, p. 306.

I	$z < x + x'$ logic		
II	$(z < x) < (z < a)$,	$(z < x') < (z < a')$	definition
III	$z < x \cdot a + x' \cdot a'$ logic		
IV	$x \equiv a'$		
V	$z < a \cdot a' + a' \cdot a$		
VI	$(z < a) < (z < a')$,	$(z < a') < (z < a)$.	

Applied to the paradox, "This proposition is false" this gives the following analysis:

I. This proposition is what its predicate says or it is not.

II. That this proposition is what its predicate says implies that it is true; and that it is not what its predicate says implies that it is false.

III. Then it is either what its predicate says and true, or not what its predicate says and false.

IV. Let its predicate say it is false.

V. Then it is either false and true, or true and false.

VI. Stating the conclusion in another way we have: That this proposition is false implies that it is true, and that it is true implies that it is false.

With this form as a starting point Rüstow constructs a theory of the paradoxes that has close resemblance to Russell's theory of types, though the only form of Russell's solution with which he was acquainted was the earlier "no-class theory" offered in the *Principles of Mathematics*. The result of the symbolic statement of the paradoxes has been to show a definitional element into which a variable term enters, the element II of the form given above. The paradox does not result until for this term there has been substituted a value that "contradicts one of the constants in the definition." A definition that contains a variable Rüstow calls *mehrdeutig*, and finds the remedy for paradox in a restriction of such definitions just as Russell finds it in a restriction of propositional functions. This restriction consists in declaring any definition containing a variable illegitimate if it is possible for the variable to take on a value that "contradicts one of the

constants in the definition." Rüstow is not so definite as Russell as to the way in which his solution is to be carried out. If we take what has been given as final the result will be merely that all ambiguous definitions are illegitimate and it is doubtful whether we have made much progress or not; for consider the proposition "This proposition is true." Here we have the same definitional element as in "This proposition is false." Is this proposition to be declared illegitimate as was the other? If it is, just as in the case of the theory of types, we have excluded propositions that offer no paradox. If we interpret Rüstow as meaning his restriction of ambiguous definitions to be a principle that forms part of every logical premise in the form, "No definition containing a variable that can take on values contradicting a constant in the definition is legitimate" we have a situation exactly like the outcome of Russell's theory. On the other hand, if we take it to mean that *This definition is valid for all values of the variable except the one that contradicts one of its constants* we have thrown no more light on the situation than to say "Paradoxical propositions are paradoxical."

Another writer who has devoted some attention to the theory of types is the mathematician Poincaré who carried on an extended discussion with Russell in the *Revue de Metaphysique et de Morale*. The substitute which Poincaré offered was not a logical theory but rather a practical rule for avoiding paradox, namely the avoidance of non-predicative definitions. Logic, he holds, is a study of the properties of classification, but a classification is not amenable to the purposes of logic unless it is a completed one. The paradoxical classes are not complete since they are always open to the addition of new members which presuppose the classification made. The definition of E as the totality of decimals definable in a finite number of words is not legitimate since the totality can never be completed. New decimals can always be added which presuppose the completion of the totality. All definitions

are classifications, and this definition is non-predicative, or this classification cannot be completed. Poincaré proposes for this particular case the amended definition of Richard which defines E as the totality of decimals definable in a finite number of words without mention of the totality E itself. Russell points out that this amended definition contains a vicious circle in a very flagrant form.[1]

Another suggestion has been made by Zermelo.[2] As Poincaré excludes definitions that are not predicative, so Zermelo would exclude those that are not *definit*. This is defined as follows: "*Eine Frage oder Aussage E über deren Gültigkeit oder Ungültigkeit die Grundbezeiungen des Bereiches, vermöge der Axiome und der allgemeingültigen logischen Gesetze ohne willkür unterscheiden, heisst definit.*" This amounts to making the starting point the class rather than the individual, and would necessitate the definition of an individual by some class to which it belonged, and of a class by some higher class of which it could be shown a part.[3] The class of all classes, the ordinal of all ordinals, the totality of all decimals definable in a finite number of words are not *definit* and so are not to be allowed.[4]

This theory of Zermelo's is really only the starting point for a theory since it does not go on to give a criterion by which we could determine when a question was decided by the fundamental relations of the range. In Russell's device we

[1] La logique de l'infini, *Rev. de M.*, 1909: 461.

[2] Untersuchungen über die Grundlagen der Mengen-lehre, I, *Math. Ann.*, 65: 261–281.

[3] D. Hilbert, Foundations of Logic and Arithmetic, Monist: XV. This is like Herbart's solution in which he uses functions such as $A(x^{(0)})$, and $A(x^{0})$, which last is Russell's (x); $f(x)$. In these functions the range of the "arbitraries" or variables is confined to "thought things that have been previously defined," so that the function itself could not occur among the possible values. Moreover he defines an aggregate as a thought thing m of which the combinations mx are elements, so that the class of all classes or the ordinal of all ordinals is impossible.

[4] This last paradox, Jourdain has pointed out, rests on an ambiguity in the word definable.

have such a criterion, and the fact that it is necessary Russell has expressed in much the same way as Zermelo. "What is necessary,[1] is not that the values (of a variable in a function) should be given individually and extensionally, but that the totality of the values should be given intensionally, so that, concerning any assigned object it is at least theoretically determinate whether or not the said object is a value of the function."

There are two proposed methods of dealing with the paradoxes through the universe of discourse which were both developed in criticism of Russell. Mr. Shearman[2] in comment on Russell's chapter in the original work on the Contradiction takes up the contradiction of the class of classes not members of themselves. He holds that the subject term in any proposition can only denote "things." The sentence "Happy is happy" can have one of three meanings; it can be "Happy things are happy," "The concept happy is happy," or noise merely. The first of these is a truism, the second is false. In the proposition which states that a class is a member of itself the mistake arises in taking a simple quality from the predicate as subject and then assuming that it is complex. The attributes determining a class cannot themselves be members of that class. The class of classes not members of themselves cannot be said to be or not to be a member of itself. The reason is that the class in its connotation is different from the class as denotation.

This is practically what is suggested by Mr. Harold Chapman Brown.[3] He maintains that the propositional functions containing an apparent variable are what the traditional logic held to be verbal propositions, and a verbal proposition must be in a different universe of discourse from the connotation of the subject term. It cannot be both a connotation and a

[1] Principia Mathematica, I: 42.

[2] Russell's Principles of Mathematics, *Mind*, XVI, 261.

[3] *Jour. Ph. Ps. Sci. Meth.*, VII, 85.

denotation in the same respects by the principle of contradiction.

Both of these suggestions along with Russell's apply as well to "This proposition is true," "This proposition has subject and predicate," and "All propositions are true or false" as to the paradoxical propositions, since all of these pretend to be at the same time connotation and denotation in that they denote themselves. If they denote themselves at all surely they denote themselves as connotation. But none of these propositions lead to paradox. The effect of the analysis that leads to the conclusion that the propositions cannot be at once connotation and denotation is substantially that of the Schoolmen's "Pars propositionis non potest supponere pro toto."

Mr. Brown is right in thinking that the theory of types involves the familiar conception of the universe of discourse, but it is not merely a rediscovery of the universe, it is rather a theory of the universe of discourse. Without some regulative principle the universe of discourse is arbitrary and lawless. Mr. Brown says *that* two things are in different universes, while the theory of types makes clear *when* two things are in different universes.

The significant fact about the paradoxes is that they do not exist until conditions amounting to definitions of the terms are introduced. Russell and Rüstow both notice that there is one value of the variable, which means that there is one definition, that results in paradox. Russell avoids the paradoxes by a principle that rules out any premise that refers to itself in one of its terms. Rüstow's more obscure provision rules out any premise that "contains a variable unless there is excluded from the range of the variable any value contradicting one of the constants of the definition." It is clear what is meant by "contradicting a constant" in the type forms given, but what this would be in general he

does not say. It cannot mean the contradictory of any constant in the definition, for definitions can be framed wherein that is done with no harm resulting. His choice of this phrase is probably due to the form in which he chose to state the paradoxes, and the paradoxes can be given symbolic statement in which no such substitution as $x = a'$ is made and the provision against it then becomes meaningless.

Russell and Rüstow both find in the paradoxes propositions that do not hold for all values of the terms. Both meet this difficulty with a general principle that legislates for all values of the terms, Russell with the principle of the vicious circle that excludes a class of values that make reference to the proposition itself, Rüstow with a principle that excludes "values that contradict a constant in the definition." Of these two Russell's restriction excludes more values than are necessary for the purpose of the restriction and makes illegitimate propositions in which was made not the troublesome substitution but a very harmless one. Rüstow avoids this disadvantage at the expense of solving a special case that the paradoxes need not resemble. We can so state any of the paradoxes that the contradiction is not introduced by giving a variable a value that is the contradictory of one of the constants in the proposition. One of the paradoxes will be stated in such a form below.

It is possible to approach the solution in another way that does not involve any departure from traditional logic. In all of the examples that have been given we find nothing paradoxical in the discovery that a proposition implies its contradictory. For this we are to some extent indebted to the logistic. That $p < p'$ is merely the criterion for the falseness of p. We may express this by: $p < 0$. If nothing were true except that certain propositions imply their own contradictories there would be no difficulty. But this is not the

whole case. The propositions under discussion seem to be implied by their own contradictories as well as to imply them. This is a true paradox for it makes $p < p'$ and $p' < p$ and results in the ordinary algebra in making 1 equivalent to 0.

That this condition exists has been left to verbal statement. To make it analytically true the proposition must be put in symbolic form and the implication shown to follow, not merely stated to follow. This involves supplying premises that are implicit in the definitions of the terms used until we have an implication that is true for all values of the terms. In other words it is not analytically true that $p < p'$. This is a mere statement. That (this proposition is false) < (this proposition is false)' or, $(a < b) < (a < b)'$ can be made logically explicit by giving a and b definitions. It is perfectly clear that it does not hold for all values of a and b as it now stands. The introduction of these definitions will make the implication explicit.

Let us introduce these necessary premises. The implication may be so stated:

$$\{[(a < b) < b] < (a < b)'\} \cdot \{(a < b) < a\} \cdot \{a < b\} < (a < b)'$$

Three premises have been necessary. The first is contained in the definition of "false," the second in the definition of "this proposition," and the third is the proposition which we desire to imply its contradictory. As the whole implication stands it is true for all values of the terms. Using "f" for "false" we may express the fact that the first premise is a definition of "false" so:

$$\{[(a < b) < b] < (a < b)'\} = (b = f)$$

If we substitute f for b this premise becomes unity. That is, it is true for the value f as defined. This may not be a complete definition of "false" but it is an element in a complete definition and may be regarded as the part of a complete definition that we need. The test of its correctness is that

there is no term save "false" that satisfies it, leaving out of consideration the values 0 and 1, since the use of either of these makes the complete proposition analytically true directly.

If we regard this first premise as the definition of f then the substitution of f for b makes it unity and it can be left unexpressed. The complete statement then becomes:

$$[(a < f) < a] \cdot [a < f] < [a < f]'$$

The term a need not be completely defined in order to make the factor "$(a < f) < a$" unity and so reduce the expression to the form, "$(a < b) < (a < b)'$." We need only introduce the restriction:

$$(p < a) < [(a < f) < a]$$

This is true whether p is taken to represent "this proposition" or "all propositions." The second premise in the original proposition we may take to be an element common to the definition of "this proposition" and "all propositions." For the solution of the paradox it is not necessary to complete the definition of either.

With the substitution of p for a we may now drop the factor "$(a < f) < a$" as a unit factor (to any implication such as "$1 \cdot 1 \cdot x < y$" the first two factors are unessential) and there remains the proposition in the desired form:

$$(p < f) < (p < f)'$$

All this goes only to make the implication of its contradictory analytically sound for "This proposition is false." This makes of the proposition merely a null-proposition like many others with which we are familiar, for any false proposition implies its contradictory. If there is to be paradox this must be accompanied by the proof that $(p < f)' < (p < f)$, but in order to reach this result it is necessary to use the term p or "this proposition" in an ambiguous sense as referring both to "this proposition is false," or $(p < f)$ and to "'this